

# ON REGULAR STEIN NEIGHBORHOODS OF A UNION OF TWO MAXIMAL TOTALLY REAL SUBSPACES IN $\mathbb{C}^n$

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**ABSTRACT.** We present a construction of regular Stein neighborhoods of a union of maximally totally real subspaces  $M = (A + iI)\mathbb{R}^n$  and  $N = \mathbb{R}^n$  in  $\mathbb{C}^n$ , provided that the entries of a real  $n \times n$  matrix  $A$  are sufficiently small. Our proof is based on a local construction of a suitable plurisubharmonic function  $\rho$  near the origin, such that the sublevel sets of  $\rho$  are strongly pseudoconvex and admit strong deformation retraction to  $M \cup N$ . We also give the application of this result to totally real immersions of real  $n$ -manifolds in  $\mathbb{C}^n$  with only finitely many double points, and such that the union of the tangent spaces at each intersection in some local coordinates coincides with  $M \cup N$ , described above.

## 1. INTRODUCTION

Many classical problems in complex analysis are solvable on Stein manifolds (see for instance [9]). It is therefore a very useful property for a subset of a manifold to have open Stein neighborhoods. However, to solve certain problems some further suitable properties of such neighborhoods are needed. For sets that either have tubular neighborhoods or allow uniformly  $H$ -convex neighborhoods one obtains holomorphic approximation theorems (see e.g. Nirenberg and Wells [11], Chirka [2]).

It is also important to control the homotopy type or the shape of the neighborhoods, and hence having the so-called *regular* neighborhoods; these are neighborhoods which admit a strong deformation retraction to a given set. By the results of Forstnerič [5, Theorem 2.2] (see also [4]) and Slapar [13] every closed real surface which is smoothly immersed into a complex surface has a basis of regular Stein neighborhoods, provided that there are only finitely many double points and only hyperbolic complex points, and they are all of special type. Near a special double point the surface is given as a model case of a union of two totally real planes in  $\mathbb{C}^2$ , intersecting only at the origin. Proposition 4.3 and [14, Proposition 4.3] by the author further extend the above result, but still only for some special cases.

In this paper we consider the generalization to higher dimensions, to the case of a union of two totally real subspaces of maximal dimension in  $\mathbb{C}^n$ , intersecting only at the origin. Every such union is complex-linearly equivalent to  $M(A) \cup N = (A + iI)\mathbb{R}^n \cup \mathbb{R}^n$ , where  $A$  is a real matrix determined up to real conjugacy and such

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that  $i$  is not an eigenvalue of  $A$ . The problem is to find a suitable plurisubharmonic function  $\rho$  near the intersection. By a result of Weinstock [15] each compact subset of  $M(A) \cup N$  is polynomially convex if and only if  $A$  has no purely imaginary eigenvalue of modulus greater than one. In this case one can easily obtain a non-negative plurisubharmonic function, vanishing on  $M(A) \cup N$ . However, to construct regular neighborhoods additional hypotheses on the gradient  $\nabla \rho$  are needed. For this reason, as in [14] we prefer to work with functions, depending polynomially in squared Euclidean distance functions to  $M$  and  $N$  respectively. Furthermore, we are now able to give more streamlined computations concerning the Levi form of  $\rho$ . This enables us to prove the existence of regular Stein neighborhoods of  $M(A) \cup N$ , provided that the eigenvalues of  $A$  are sufficiently close to zero; see Theorem 4.2 for an estimate of how small these eigenvalues can be. At the end we also give the application of this result to totally real immersions of real  $n$ -manifolds in  $\mathbb{C}^n$  with only finitely many double points, and such that the union of the tangent spaces at each intersection in some local coordinates coincides with  $M(A) \cup N$ , described above. In connection to this we also note that Weinstock's result has been recently generalised by Gorai [6] and Shafikov and Sukhov [12, Theorems 1.3 and 4.2], to the effect that a union of two maximally totally real submanifolds in  $\mathbb{C}^n$ , intersecting transversally at the origin, is polynomially convex near the origin, if the union of their tangent spaces at the origin is polynomially convex near the origin.

## 2. THE EUCLIDEAN DISTANCE FUNCTION TO A TOTALLY REAL SUBSPACE

Throughout this paper  $z = (z_1, \dots, z_n)$  will be standard holomorphic coordinates and  $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$  corresponding real coordinates on  $\mathbb{C}^n = (\mathbb{R} + i\mathbb{R})^n \approx \mathbb{R}^{2n}$  with respect to  $z_j = x_j + iy_j$  for all  $j \in \{1, \dots, n\}$ .

By  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  respectively we denote the Euclidean inner product and Euclidean distance on any  $\mathbb{C}^r$ ,  $r \in \mathbb{N}$ :

$$\begin{aligned} \langle \xi, \eta \rangle &= \sum_{j=1}^r \xi_j \bar{\eta}_j, & \xi &= (\xi_1, \dots, \xi_r), & \eta &= (\eta_1, \dots, \eta_r), \\ |\xi| &= \sqrt{\langle \xi, \xi \rangle} \end{aligned}$$

In real notation with  $\xi_j = s_j + it_j$  and  $\eta_j = u_j + v_j$  for  $j \in \{1, \dots, r\}$  we have

$$\langle \xi, \eta \rangle = \sum_{j=1}^r (s_j u_j + t_j v_j), \quad \xi = (s_1, \dots, s_r, t_1, \dots, t_r), \quad \eta = (u_1, \dots, u_r, v_1, \dots, v_r).$$

Next, recall that a real linear subspace in  $\mathbb{C}^n$  is called *totally real* if it contains no complex subspace. Let now  $M$  and  $N$  be linear totally real subspaces of maximal dimension  $n$  in  $\mathbb{C}^n$ , intersecting at the origin. It is not difficult to prove (see e.g. [15]) that there exists a non-singular complex linear transformation which maps  $N$  onto  $\mathbb{R}^n \approx (\mathbb{R} + i0)^n \subset \mathbb{C}^n$  and  $M$  onto  $M(A) = (A + iI)\mathbb{R}^n$ , where  $A$  is the real Jordan canonical form, i.e.  $A$  is a square block matrix, having zero-matrices as off-diagonal blocks, and each of the main diagonal blocks satisfies one of the two conditions listed below:

- (1) A matrix with  $a \in \mathbb{R}$  on the main diagonal, possibly with  $\delta \in \mathbb{R}\{0\}$  on the upper diagonal, and zeros otherwise, i.e.

$$(2.1) \quad \begin{bmatrix} a & \delta & & \\ & a & \ddots & \\ & & \ddots & \delta \\ & & & a \end{bmatrix},$$

- (2) A square block matrix, having  $2 \times 2$  main diagonal blocks with complex eigenvalues, possibly with the  $2 \times 2$  identity-matrix  $I_2$  multiplied by  $\delta \in \mathbb{R}\{0\}$  on the upper diagonal, and  $2 \times 2$  zero-matrices otherwise, i.e.

$$(2.2) \quad \begin{bmatrix} C & \delta I_2 & & \\ & C & \ddots & \\ & & \ddots & \delta I_2 \\ & & & C \end{bmatrix}, \quad C = \begin{bmatrix} c & -b \\ b & c \end{bmatrix}, \quad b, c \in \mathbb{R}, \quad c \neq 0, \quad c^2 + (1 - b^2)^2 \neq 0,$$

Moreover,  $A$  is uniquely determined up to the order of diagonal blocks, and a non-zero real number  $\delta$  can be chosen arbitrarily. Also, the degerate case (i.e.  $1 \times 1$  matrix in case (2.1) and  $2 \times 2$  matrix in case (2.2)) is considered here, though it lacks an upper diagonal or block-upper diagonal, respectively. Clearly,  $A$  is diagonalizable if and if all diagonal blocks are degenerate.

For any  $\delta \in \mathbb{R}$  we set

$$(2.3) \quad A_\delta = \text{diag}(A_{\delta,1}, \dots, A_{\delta,\alpha})$$

to be a  $n \times n$  block diagonal matrix and such that for every  $j \in \{1, \dots, \alpha\}$  the diagonal block  $A_{\delta,\alpha} \in \mathbb{R}^{n_j \times n_j}$  is of the form (2.1) or (2.2), possibly degenerate, and  $n_1 + \dots + n_\alpha = n$ .

**Lemma 2.1.** *Let  $A_\delta$  for  $\delta \in \mathbb{R}$  be defined as in (2.3), and let  $d_{M(A_\delta)}(x, y)$  be the squared Euclidean distance from  $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$  to  $M(A_\delta)$  in  $\mathbb{R}^{2n}$ . If  $A_\delta$  is non-diagonalizable for  $\delta \neq 0$ , then*

$$d_{M(A_\delta)}(x, y) = d_{M(A_0)}(x, y) + q_\delta(x, y),$$

where  $q_\delta$  is a homogeneous polynomial of degree 2 in  $x, y$  and such that its coefficients are rational functions in  $\delta$  and they have a zero at  $\delta = 0$ . Furthermore, if

$A_0 = \text{diag}(C_1, C_2, \dots, C_\beta, a_1, \dots, a_\gamma)$ ,  $2\beta + \gamma = n$ , where  $C_j = \begin{bmatrix} c_j & -b_j \\ b_j & c_j \end{bmatrix}$ , and  $c_1, b_1, \dots, c_\beta, b_\beta, a_1, \dots, a_\gamma \in \mathbb{R}$ , then

$$(2.4) \quad d_{M(A_0)}(x, y) = \sum_{j=1}^{\beta} \frac{(x_{2j-1} - c_j y_{2j-1} + b_j y_{2j})^2 + (x_{2j} - c_j y_{2j} - b_j y_{2j-1})^2}{1 + c_j^2 + b_j^2} \\ + \sum_{j=2\beta+1}^{2\beta+\gamma} \frac{(x_j - a_j y_j)^2}{1 + a_j^2}.$$

In particular, if  $A_0 = \text{diag}(C_1, C_2, \dots, C_{n/2})$  (respectively  $A_0 = \text{diag}(a_1, \dots, a_n)$ ), then  $d_{M(A_0)}(x, y)$  is equal to the first (respectively second) term of the above sum (2.4) for  $\beta = n/2$  (respectively  $\beta = 0, \gamma = n$ ).

*Proof.* Let  $M(A_{\delta,j})^\perp$  be the orthogonal complement of  $M(A_{\delta,j})$  in  $\mathbb{C}^{n_j}$  with respect to the standard inner product. It gives decompositions of  $M(A_\delta)$  and its orthogonal complement  $M(A_\delta)^\perp$  respectively into pairwise orthogonal linear subspaces:

$$M(A_\delta) = M_{\delta,1} \oplus \dots \oplus M_{\delta,\alpha}, \quad M(A_\delta)^\perp = M_{\delta,1}^\perp \oplus \dots \oplus M_{\delta,\alpha}^\perp,$$

where  $M_{\delta,j} = \{0\}^{n_1+\dots+n_{j-1}} \times M(A_{\delta,j}) \times \{0\}^{n_{j+1}+\dots+n_\alpha}$  and  $M_{\delta,j}^\perp = \{0\}^{n_1+\dots+n_{j-1}} \times M(A_{\delta,j})^\perp \times \{0\}^{n_{j+1}+\dots+n_\alpha}$  for all  $j$ .

If for any linear subspace  $M \subset \mathbb{C}^r$  we denote the squared Euclidean distance and squared Euclidean projection to  $M$  in  $\mathbb{C}^r$  respectively by  $d_M$  and  $p_M$ , we obtain  $d_{M(A_\delta)} = p_{M(A_\delta)^\perp} = \sum_{j=1}^\alpha p_{M_{\delta,j}^\perp} = \sum_{j=1}^\alpha d_{M_{\delta,j}} = \sum_{j=1}^\alpha d_{M(A_{\delta,j})}$ . For this reason, it is sufficient to prove the lemma for the case when  $A_\delta$  is of the form (2.1) or (2.2).

First, we consider the case when  $A_\delta \in \mathbb{R}^{n \times n}$  is of the form (2.1). This implies that  $M(A_\delta)$  is given as a span of  $n$  linearly independent vectors

$$(2.5) \quad M(A_\delta) = \text{Span}\{f_j + ae_j + \delta e_{j-1}\}_{2 \leq j \leq n} \cup \{ae_1 + f_2\},$$

where  $\{e_1, f_1, \dots, e_n, f_{2n}\}$  is the standard ortho-normal basis of  $\mathbb{R}^{2n}$ . We observe further that the orthogonal complement is then equal to

$$(2.6) \quad M(A_\delta)^\perp = \text{Span}\{e_j - af_j - \delta f_{j+1}\}_{1 \leq j \leq n-1} \cup \{e_n - af_n\}$$

Indeed, since every  $e_j$  for  $j \in \{1, \dots, n\}$  is orthogonal to all but one vector in the span (2.6), the span contains  $n$  linearly independent vectors.

To simplify the computations we denote the vectors in (2.6) by

$$g_j = e_j - af_j - \delta f_{j+1}, \quad 1 \leq j \leq n-1, \quad g_n = e_n - af_n$$

and we perform Gram-Schmidt process  $g'_m = g_m - \sum_{j < m} \frac{\langle g_m, g'_j \rangle}{|g'_j|^2} g'_j$  to obtain orthogonal basis  $\{g'_1, \dots, g'_n\}$  of  $M(A_\delta)^\perp$ . We show by induction that

$$(2.7) \quad g'_j = e_j - af_j - \delta v_j, \quad j \in \{1, \dots, n\},$$

where the components of  $v_j$  are rational functions in  $\delta$  and without a pole at  $\delta = 0$ . Suppose (2.7) holds for all  $1 \leq j < m$ . For any  $j < m$  it then follows that  $|g'_j|^2$  is a rational function in  $\delta$  and it has no zero at  $\delta = 0$ , and  $\langle g_m, g'_j \rangle$  is a rational function in  $\delta$  and without a pole at  $\delta = 0$ . This immediately implies (2.7).

The squared Euclidean distance of  $(x, y)$  to  $M(A_\delta)$  is thus

$$\begin{aligned} d_{M(A_\delta)}(x, y) &= \sum_{j=1}^n \frac{\langle g'_j, (x, y) \rangle^2}{|g'_j|^2} = \sum_{j=1}^n \frac{\left( \langle e_j - af_j, (x, y) \rangle + \delta \langle v_j, (x, y) \rangle \right)^2}{|e_j - af_j - \delta v_j|^2} \\ &= \sum_{j=1}^n \frac{\langle e_j - af_j, (x, y) \rangle^2}{|e_j - af_j|^2} + \delta \sum_{j=1}^n \frac{\langle v_j, (x, y) \rangle \langle 2g'_j + 3\delta v_j, (x, y) \rangle}{|g'_j|^2} \\ &\quad - \delta \sum_{j=1}^n \frac{\langle e_j - af_j, (x, y) \rangle^2 \langle 2g'_j + 3\delta v_j, v_j \rangle}{|e_j - af_j|^2 |g'_j|^2}. \end{aligned}$$

Observe that the sums in the last are homogeneous polynomials of degree 2 in  $x, y$  and such that its coefficients are rational functions in  $\delta$  and in addition they have no a pole at  $\delta = 0$ . Further, for  $\delta = 0$  in (2.6) we see that  $M(A_0)^\perp =$

$\text{Span}\{e_j - af_j\}_{1 \leq j \leq n}$  is a span of orthogonal vectors, hence the first term in the above sum is equal to

$$d_{M(A_0)}(x, y) = \sum_{j=1}^n \frac{(x_j - ay_j)^2}{1 + a^2}.$$

This completes proof in the case when  $A_\delta$  is of the form (2.1).

In a similar fashin we shall now deal with the case when  $A_\delta$  is of the form (2.2) and thus  $n$  even. We have

$$M(A_\delta) = \text{Span}\{ce_j + f_j + be_{j+1} + \delta e_{j-2}, -be_j + f_{j+1} + cf_{j+1} + \delta e_{j-1}\}_{j \in \{3, 5, \dots, n-1\}} \\ \cup \{ce_1 + f_1 + be_2, -be_1 + f_2 + ce_2\}$$

and

$$M(A_\delta)^\perp = \text{Span}\{e_j - cf_j + bf_{j+1} - \delta f_{j+2}, e_{j+1} - bf_j - cf_{j+1} - \delta e_{j+3}\}_{j \in \{1, 3, \dots, n-3\}} \\ (2.8) \quad \cup \{-cf_{n-1} + e_{n-1} + bf_n, -bf_{n-1} + e_n - cf_n\}.$$

Again, every  $e_j$  for  $j \in \{1, \dots, n\}$  is orthogonal to all but one vector in the span (2.8), hence the vectors in the span are linearly independent.

Next, we set

$$h_j = -cf_j + e_j + bf_{j+1} - \delta f_{j+2}, \quad j \in \{1, 3, \dots, n-3\} \\ h_j = -bf_{j-1} + e_j - cf_j - \delta e_{j+3}, \quad j \in \{2, 4, \dots, n-2\} \\ h_{n-1} = -cf_{n-1} + e_{n-1} + bf_n, \quad h_{2n} = -bf_{n-1} + e_n - cf_n$$

and proceed with the Gram-Schmidt process to obtain orthogonal basis  $\{h'_1, \dots, h'_n\}$  of  $M(A_\delta)^\perp$ . Similar to (2.7), we now have

$$(2.9) \quad h'_j = -cf_j + e_j + bf_{j+1} - \delta w_j, \quad j \in \{1, 3, \dots, n-1\} \\ h'_j = -bf_{j-1} + e_j - cf_j - \delta w_j, \quad j \in \{2, 4, \dots, n\}$$

where the components of  $w_j$  are rational functions in  $\delta$  and without a pole at  $\delta = 0$ .

It follows that

$$d_{M(A_\delta)}(x, y) = \sum_{j=1, 3, \dots, n-1} \frac{\langle -cf_j + e_j + bf_{j+1}, (x, y) \rangle^2}{|-cf_j + e_j + bf_{j+1}|^2} \\ + \sum_{j=2, 4, \dots, n} \frac{\langle -bf_{j-1} + e_j - cf_j, (x, y) \rangle^2}{|-bf_{j-1} + e_j - cf_j|^2} \\ - \delta \sum_{j=1, 3, \dots, n-1} \frac{\langle -cf_j + e_j + bf_{j+1}, (x, y) \rangle^2 \langle 2h'_j + 3\delta w_j, w_j \rangle}{|-cf_j + e_j + bf_{j+1}|^2 |h'_j|^2} \\ - \delta \sum_{j=2, 4, \dots, n} \frac{\langle -bf_{j-1} + e_j - cf_j, (x, y) \rangle^2 \langle 2h'_j + 3\delta w_j, w_j \rangle}{|-bf_{j-1} + e_j - cf_j|^2 |h'_j|^2} \\ + \delta \sum_{j=1}^n \frac{\langle w_j, (x, y) \rangle \langle 2h'_j + 3\delta w_j, (x, y) \rangle}{|h'_j|^2}$$

Again, we observe that the sums in the last three terms are polynomials of degree 2 in  $x, y$  and such that its coefficients are rational functions in  $\delta$  and in addition

they have no pole at  $\delta = 0$ . Since (2.9) for  $\delta = 0$  is a span of orthogonal vectors, the first two terms in the above sum are equal to

$$\begin{aligned} d_{M(A_0)}(x, y) &= \sum_{j=1,3,\dots,n-1} \frac{(x_j - cy_j + by_{j+1})^2}{1 + c^2 + b^2} + \sum_{j=2,4,\dots,n} \frac{(x_j - cy_j - by_{j-1})^2}{1 + c^2 + b^2} \\ &= \sum_{j=1}^n \left( \frac{(x_{2j-1} - cy_{2j-1} + by_{2j})^2}{1 + c^2 + b^2} + \frac{(x_{2j} - cy_{2j} - by_{2j-1})^2}{1 + c^2 + b^2} \right) \end{aligned}$$

This completes proof of the lemma.  $\square$

### 3. LOCAL CONSTRUCTION AT THE INTERSECTION

Given a  $\mathcal{C}^2$ -function  $f: \Omega \rightarrow \mathbb{R}$  on an open set  $\Omega \subset \mathbb{C}^n$  we denote holomorphic and antiholomorphic derivatives of  $f$  by  $\frac{\partial f}{\partial z_j} = \frac{1}{2} \left( \frac{\partial f}{\partial x_j} - i \frac{\partial f}{\partial y_j} \right)$ ,  $\frac{\partial u}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial u}{\partial x_j} + i \frac{\partial u}{\partial y_j} \right)$ . For  $1 \leq r \leq n$  we further introduce the notation

$$\left( \frac{\partial f}{\partial z} \right)_r = \left( \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_r} \right), \quad \left( \frac{\partial f}{\partial \bar{z}} \right)_r = \left( \frac{\partial f}{\partial \bar{z}_1}, \dots, \frac{\partial f}{\partial \bar{z}_r} \right),$$

The *Levi form* is defined by

$$\mathcal{L}_{(z)}(f; \xi) = \sum_{j,k=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(z) \xi_j \bar{\xi}_k, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n.$$

A function  $u$  is *strictly plurisubharmonic* if and only if  $\mathcal{L}_{(z)}(f; \cdot)$  is a positive definite Hermitian quadratic form at each point  $z \in \Omega$ , and this is the case precisely when all leading principal minors of the complex Hessian matrix of  $f$  are positive on  $\Omega$ , i.e. the determinant of

$$H_r^{\mathbb{C}}(f) = \left[ \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^r$$

is positive for every  $r \in \{1, 2, \dots, n\}$ . In particular,  $H_n^{\mathbb{C}}(f)$  is the complex Hessian matrix of  $f$ .

To simplify the computations of the complex Hessian matrices or its determinants in the proceedings of this section, let us also introduce the following matrix notation:

$$\begin{aligned} (3.1) \quad \xi \eta^T &= \sum_{j=1}^r \xi_j \eta_j, \quad \xi = (\xi_1, \dots, \xi_r), \quad \eta = (\eta_1, \dots, \eta_r), \\ \xi^T \eta &= [\xi_j \eta_k]_{j,k=1}^r. \end{aligned}$$

Before starting with the complex Hessians of functions, which are related to totally real subspaces, we prove a simple fact on the determinant of a sum of a matrix and some matrices of rank one. It is stated in the above notation.

**Lemma 3.1.** *Let  $u_1, \dots, u_s, v_1, \dots, v_s \in \mathbb{C}^2$ , where  $s \in \mathbb{N}$ , and let  $B \in \mathbb{C}^{2 \times 2}$ . Then*

$$\begin{aligned} \det(B + \sum_{l=1}^s u_l v_l^T) &= \det(B) + \text{Tr}(B) \sum_{l=1}^s v_l^T u_l - \sum_{l=1}^s v_l^T B u_l + \det \left( \sum_{l=1}^s u_l v_l^T \right), \\ \det \left( \sum_{l=1}^s u_l v_l^T \right) &= \frac{1}{2} \sum_{l,m=1}^s (u_l^T v_l u_m^T v_m - u_l^T v_m u_m^T v_l). \end{aligned}$$

*Proof.* We prove the lemma by induction on  $s$ . The case  $s = 1$  is a simple matrix identity

$$(3.2) \quad \det(B + uv^T) = \det(B) + \text{Tr}(B)u^T v - v^T B u, \quad B \in \mathbb{C}^{2 \times 2}, \quad u, v \in \mathbb{C}^2.$$

Next, we assume that the statement holds for some  $s \in \mathbb{N}$ . Applying (3.2), we then get

$$\begin{aligned} \det\left(B + \sum_{l=1}^{s+1} u_l v_l^T\right) &= \det\left(B + \sum_{l=1}^s u_l v_l^T\right) + \text{Tr}\left(B + \sum_{l=1}^s u_l v_l^T\right) u_{s+1}^T v_s \\ &\quad - v_{s+1}^T \left(B + \sum_{l=1}^s u_l v_l^T\right) u_{s+1} \\ &= \det(B) + \text{Tr}(B) \sum_{l=1}^s v_l^T u_l - \sum_{l=1}^s v_l^T B u_l + \frac{1}{2} \sum_{l,m=1}^s (u_l^T v_l u_m^T v_m - u_l^T v_m u_m^T v_l) \\ &\quad + \text{Tr}(B) u_{s+1}^T v_s + \sum_{l=1}^s u_l^T v_l u_{s+1}^T v_s - v_{s+1}^T B u_{s+1} - v_{s+1}^T \sum_{l=1}^s u_l v_l^T u_{s+1} \end{aligned}$$

Regrouping the like terms now easily concludes the proof.  $\square$

As in [14] we prefer to work with polynomials in squared Euclidean distance functions to maximal totally real subspaces in  $\mathbb{C}^n$ . The following lemma is a preparation for our key result, Lemma 3.3.

**Lemma 3.2.** *Let  $A$  be a diagonalizable real  $n \times n$  matrix, and let  $d_M$  and  $d_N$  respectively be the squared Euclidean distance functions to  $M = (A + iI)\mathbb{R}^n$  and  $N = \mathbb{R}^2$ . Assume further that  $P \in \mathbb{R}[u, v]$  is a polynomial in two variables and set  $\Delta = \frac{1}{2}(\frac{\partial P}{\partial u}(d_M, d_N) + \frac{\partial P}{\partial v}(d_M, d_N))$ . Then the function*

$$\rho = P(d_M, d_N).$$

*has the following properties:*

$$(1) \quad \det(H_1^{\mathbb{C}}(\rho)) = \Delta + \frac{\partial^2 P}{\partial u^2}(d_M, d_N) \left| \frac{\partial d_M}{\partial z_1} \right|^2 + \frac{\partial^2 P}{\partial v^2}(d_M, d_N) \left| \frac{\partial d_N}{\partial z_1} \right|^2 \\ + 2 \frac{\partial^2 P}{\partial u \partial v}(d_M, d_N) \text{Re} \left\langle \frac{\partial d_M}{\partial z_1}, \frac{\partial d_N}{\partial z_1} \right\rangle.$$

(2) *If  $A$  is a real diagonal matrix, then for every  $r \in \{2, \dots, n\}$  it follows that*

$$(3.4) \quad \det(H_r^{\mathbb{C}}(\rho)) = \Delta^r + \Delta^{r-1} \left( \frac{\partial^2 P}{\partial u^2}(d_M, d_N) \left| \left( \frac{\partial d_M}{\partial z} \right)_r \right|^2 + \frac{\partial^2 P}{\partial v^2}(d_M, d_N) \left| \left( \frac{\partial d_N}{\partial z} \right)_r \right|^2 \right) \\ + 2 \Delta^{r-1} \frac{\partial^2 P}{\partial u \partial v}(d_M, d_N) \text{Re} \left\langle \left( \frac{\partial d_M}{\partial z} \right)_r, \left( \frac{\partial d_N}{\partial z} \right)_r \right\rangle \\ + \Delta^{r-2} \det(H^{\mathbb{R}}(P)_{(d_M, d_N)}) \sum_{1 \leq j < k \leq r} Z_{jk}(A),$$

where  $Z_{jk}(A) = \left( \sum_{m=j,k} \left| \frac{\partial d_M}{\partial z_m} \right|^2 \sum_{m=j,k} \left| \frac{\partial d_N}{\partial z_m} \right|^2 - \left| \sum_{m=j,k} \frac{\partial d_M}{\partial z_m} \frac{\partial d_N}{\partial \bar{z}_m} \right|^2 \right)$  and  $H^{\mathbb{R}}(P)$  is the real Hessian of  $P$ . Furthermore, if  $r = n = 2$ , we have  $\left| \left( \frac{\partial d_M}{\partial z} \right)_2 \right|^2 = d_M$ ,  $\left| \left( \frac{\partial d_N}{\partial z} \right)_2 \right|^2 = d_N$  and  $Z_{12}(A) = d_M d_N - \left| \sum_{j=1,2} \frac{\partial d_M}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_j} \right|^2$ .

(3) If  $n = 2$  and  $A = \begin{bmatrix} c & -b \\ b & c \end{bmatrix}$  with  $b, c \in \mathbb{R}$ , then

$$\begin{aligned}
 \det(H^{\mathbb{C}}(\rho)) &= \Delta \left( \frac{\partial^2 P}{\partial u^2}(d_M, d_N) d_M + \frac{\partial^2 P}{\partial v^2}(d_M, d_N) d_N + 2 \frac{\partial^2 P}{\partial u \partial v}(d_M, d_N) \operatorname{Re} \left( \sum_{j=1,2} \frac{\partial d_M}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_j} \right) \right) \\
 (3.5) \quad &+ \Delta^2 + \det(H_2^{\mathbb{R}}(P)_{(d_M, d_N)}) \left( d_M d_N - \left| \sum_{j=1,2} \frac{\partial d_M}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_j} \right|^2 \right) \\
 &- \left( \frac{b}{1+b^2+c^2} \frac{\partial P}{\partial u}(d_M, d_N) \right)^2 - \frac{2b^2}{1+b^2+c^2} d_M \frac{\partial^2 P}{\partial u^2}(d_M, d_N) \frac{\partial P}{\partial u}(d_M, d_N).
 \end{aligned}$$

*Proof.* A straightforward computation yields

$$\begin{aligned}
 \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} &= \frac{\partial^2 P}{\partial u^2}(d_M, d_N) \frac{\partial d_M}{\partial z_j} \frac{\partial d_M}{\partial \bar{z}_k} + \frac{\partial^2 P}{\partial v^2}(d_M, d_N) \frac{\partial d_N}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_k} \\
 &+ \frac{\partial P}{\partial u \partial v}(d_M, d_N) \left( \frac{\partial d_M}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_k} + \frac{\partial d_N}{\partial z_j} \frac{\partial d_M}{\partial \bar{z}_k} \right) \\
 &+ \frac{\partial P^2}{\partial u}(d_M, d_N) \frac{\partial d_M}{\partial z_j \partial \bar{z}_k} + \frac{\partial^2 P}{\partial v}(d_M, d_N) \frac{\partial d_N}{\partial z_j \partial \bar{z}_k}
 \end{aligned}$$

The submatrix of the complex Hessian matrix of  $\rho$ , which consists of the entries in the first  $r$  rows and in the first  $r$  columns for  $r \in \{1, 2, \dots, n\}$ , can hence in matrix notation (see (3.1)) be written as

$$\begin{aligned}
 H_r^{\mathbb{C}}(\rho) &= \frac{\partial^2 P}{\partial u^2}(d_M, d_N) \left( \frac{\partial d_M}{\partial z} \right)_r \left( \frac{\partial d_M}{\partial \bar{z}} \right)_r^T + \frac{\partial^2 P}{\partial v^2}(d_M, d_N) \left( \frac{\partial d_N}{\partial z} \right)_r \left( \frac{\partial d_N}{\partial \bar{z}} \right)_r^T \\
 (3.6) \quad &+ \frac{\partial^2 P}{\partial u \partial v}(d_M, d_N) \left( \left( \frac{\partial d_M}{\partial z} \right)_r \left( \frac{\partial d_N}{\partial \bar{z}} \right)_r^T + \left( \frac{\partial d_N}{\partial z} \right)_r \left( \frac{\partial d_M}{\partial \bar{z}} \right)_r^T \right) \\
 &+ \frac{\partial P}{\partial u}(d_M, d_N) \left[ \frac{\partial^2 d_M}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^r + \frac{\partial P}{\partial v}(d_M, d_N) \left[ \frac{\partial^2 d_N}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^r.
 \end{aligned}$$

Since  $N = \mathbb{R}^n$  we immediately get

$$(3.7) \quad d_N(x, y) = \sum_{j=1}^n y_j^2, \quad \frac{\partial d_N}{\partial z_j}(x, y) = -iy_j, \quad j \in \{1, \dots, n\}.$$

We proceed by proving property (2). In this case  $A = \operatorname{diag}(a_1, \dots, a_n)$  for  $a_1, \dots, a_n \in \mathbb{R}$  is a real diagonal matrix. Lemma 2.1 and a simple computation now yield

$$(3.8) \quad d_M(x, y) = \sum_{j=1}^n \frac{x_j - a_j y_j}{1 + a_j^2}, \quad \frac{\partial d_M}{\partial z_j}(x, y) = \frac{(1 + ia_j)(x_j - a_j y_j)}{1 + a_j^2}, \quad j \in \{1, \dots, n\}.$$

Further, (3.7) and (3.8) easily imply

$$\left[ \frac{\partial^2 d_M}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^r = \left[ \frac{\partial^2 d_N}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^r = \frac{1}{2} I_r,$$

where  $I_r$  is the  $r \times r$  identity-matrix. For  $\Delta = \frac{1}{2}(\frac{\partial P}{\partial u}(d_M, d_N) + \frac{\partial P}{\partial v}(d_M, d_N))$  we have

$$H_r^{\mathbb{C}}(\rho) = \Delta I_r + L_r,$$



where

$$(3.9) \quad L_r = \frac{\partial^2 P}{\partial u^2}(d_M, d_N) \left( \frac{\partial d_M}{\partial z} \right)_r \left( \frac{\partial d_M}{\partial \bar{z}} \right)_r^T + \frac{\partial^2 P}{\partial v^2}(d_M, d_N) \left( \frac{\partial d_N}{\partial z} \right)_r \left( \frac{\partial d_N}{\partial \bar{z}} \right)_r^T \\ + \frac{\partial^2 P}{\partial u \partial v}(d_M, d_N) \left( \left( \frac{\partial d_M}{\partial z} \right)_r \left( \frac{\partial d_N}{\partial \bar{z}} \right)_r^T + \left( \frac{\partial d_N}{\partial z} \right)_r \left( \frac{\partial d_M}{\partial \bar{z}} \right)_r^T \right)_r^T$$

Observe that for  $2 \leq r \leq n$  the matrix  $L_r$  is of rank two, hence its possible minors of order greater than 2 must be zero. Further, it is a well known fact about characteristic polynomial of the  $r \times r$  matrix which states that for any  $r \times r$  matrix  $B$  we have

$$(3.10) \quad \det(B + \lambda I_r) = \lambda^r + c_{r-1}\lambda^{r-1} + \dots + c_1\lambda + c_0, \lambda \in \mathbb{R}$$

where the coefficient  $c_j$  for  $j \in \{1, \dots, r-1\}$  is equal to the sum of all principal minors of the matrix  $B$  of order  $r-j$ . Therefore

$$(3.11) \quad \det(H_r^{\mathbb{C}}(\rho)) = \Delta^r + \text{Tr}(L_r)\Delta^{r-1} + \Delta^{r-2} \sum_{1 \leq j < k \leq r} \text{Minor}_{j,k}(L_r)$$

where  $\text{Minor}_{j,k}(L_r)$  is the corresponding principal minor of the matrix  $L_r$  with respect to the  $j$ -th and  $k$ -th row and column; this is the determinant of the submatrix of  $L_r$ , formed by taking the elements in the  $j$ -th or the  $k$ -th column, and in the  $j$ -th or the  $k$ -th row.

Using matrix notation (3.1) we can write

$$(3.12) \quad \text{Tr}(L_r) = \frac{\partial^2 P}{\partial u^2}(d_M, d_N) \left( \frac{\partial d_M}{\partial z} \right)_r \left( \frac{\partial d_M}{\partial \bar{z}} \right)_r^T + \frac{\partial^2 P}{\partial v^2}(d_M, d_N) \left( \frac{\partial d_N}{\partial z} \right)_r \left( \frac{\partial d_N}{\partial \bar{z}} \right)_r^T \\ + \frac{\partial^2 P}{\partial u \partial v}(d_M, d_N) \left( \left( \frac{\partial d_M}{\partial z} \right)_r \left( \frac{\partial d_N}{\partial \bar{z}} \right)_r^T + \left( \frac{\partial d_N}{\partial z} \right)_r \left( \frac{\partial d_M}{\partial \bar{z}} \right)_r^T \right)_r^T.$$

Also, we apply Lemma 3.1 for the case where  $B = 0$  and the sum of rank-one matrices is equal to  $L_2$ . By regrouping the terms we get

$$(3.13) \quad \text{Minor}_{j,k}(L_r) = \left( \frac{\partial^2 P}{\partial u^2}(d_M, d_N) \frac{\partial^2 P}{\partial v^2}(d_M, d_N) - \left( \frac{\partial^2 P}{\partial u \partial v}(d_M, d_N) \right)^2 \right) \\ \cdot \left( \left( \frac{\partial d_M}{\partial z_j}, \frac{\partial d_M}{\partial \bar{z}_k} \right)^T \left( \frac{\partial d_M}{\partial \bar{z}_j}, \frac{\partial d_M}{\partial z_k} \right) \left( \frac{\partial d_N}{\partial z_j}, \frac{\partial d_N}{\partial \bar{z}_k} \right)^T \left( \frac{\partial d_N}{\partial \bar{z}_j}, \frac{\partial d_N}{\partial z_k} \right) - \left| \left( \frac{\partial d_M}{\partial z_j}, \frac{\partial d_M}{\partial \bar{z}_k} \right)^T \left( \frac{\partial d_N}{\partial \bar{z}_j}, \frac{\partial d_N}{\partial z_k} \right) \right|^2 \right)$$

Finally, (3.4) now follows immediately from (3.11), (3.12), (3.13). To conclude the proof of (2), we set  $n = r = 2$  and using (3.7), (3.8), we see that

$$(3.14) \quad \left| \left( \frac{\partial d_M}{\partial z} \right)_2 \right|^2 = \sum_{j=1}^2 \left| \frac{\partial d_M}{\partial z_j} \right|^2 = d_M, \quad \left| \left( \frac{\partial d_N}{\partial z} \right)_2 \right|^2 = \sum_{j=1}^2 \left| \frac{\partial d_N}{\partial z_j} \right|^2 = d_N.$$

Next, we prove (3). By (2.1) we now have

$$(3.15) \quad d_M(x_1, y_1, x_2, y) = \frac{(x_1 - cy_1 + by_2)^2}{1 + b^2 + c^2} + \frac{(x_2 - cy_2 - by_1)^2}{1 + b^2 + c^2}.$$

Next, a simple computation shows that for any  $j \in \{1, \dots, n\}$ :

$$(3.16) \quad \frac{\partial d_M}{\partial z_1} = \frac{1+ic}{1+b^2+c^2}(x_1 - cy_1 + by_2) + \frac{ib}{1+b^2+c^2}(x_2 - cy_2 - by_1), \\ \frac{\partial d_M}{\partial z_2} = \frac{1+ic}{1+b^2+c^2}(x_2 - cy_2 - by_1) - \frac{ib}{1+b^2+c^2}(x_1 - cy_1 + by_2)$$

and

$$\left[ \frac{\partial d_M}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^2 = \begin{bmatrix} \frac{1}{2} & -\frac{ib}{1+b^2+c^2} \\ \frac{ib}{1+b^2+c^2} & \frac{1}{2} \end{bmatrix},$$

Remember that  $\Delta = \frac{1}{2}(\frac{\partial P}{\partial u}(d_M, d_N) + \frac{\partial P}{\partial v}(d_M, d_N))$  and by setting  $\varepsilon = \frac{b}{1+b^2+c^2} \frac{\partial P}{\partial u}(d_M, d_N)$ , we have

$$H^{\mathbb{C}}(\rho) = \Delta I_2 + i\varepsilon K_2 + L_2,$$

where  $L_2$  is as in (3.9) and  $K_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

From (3.10) for  $B = i\varepsilon K_2 + L_2$  and  $\lambda = \Delta$ , and using the fact  $\text{Tr}(i\varepsilon K_2 + L_2) = \text{Tr}(L_2)$ , it follows that

$$(3.17) \quad \det(H_2^{\mathbb{C}}(\rho)) = \Delta^2 + \Delta \text{Tr}(L_2) + \det(i\varepsilon K_2 + L_2).$$

It is easy to see that

$$\begin{aligned} \left( \frac{\partial d_M}{\partial z} \right)_2^T K_2 \left( \frac{\partial d_M}{\partial \bar{z}} \right)_2 &= 2ib d_M, & \left( \frac{\partial d_N}{\partial z} \right)_2^T K_2 \left( \frac{\partial d_N}{\partial \bar{z}} \right)_2 &= 0, \\ \left( \frac{\partial d_N}{\partial z} \right)_2^T K_2 \left( \frac{\partial d_M}{\partial \bar{z}} \right)_2 + \left( \frac{\partial d_M}{\partial z} \right)_2^T K_2 \left( \frac{\partial d_N}{\partial \bar{z}} \right)_2 &= 0, \end{aligned}$$

Thus, applying Lemma 3.1 for  $B = K_2$ , we obtain

$$\det(i\varepsilon K_2 + L_2) = -\varepsilon^2 - 2 \frac{\partial^2 P}{\partial (d_M, d_N)} b \varepsilon d_M + \det(L_2).$$

Since (3.14) holds also in case when  $d_M$  of the form (3.15), then by using (3.13) for  $j = 1, k = 2$  we obtain that

$$\det(L_2) = \text{Minor}_{1,2}(L_2) = \det(H_2^{\mathbb{R}}(P)_{(d_M, d_N)}) \left( d_M d_N - \left| \sum_{j=1,2} \frac{\partial d_M}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_j} \right|^2 \right).$$

Using also (3.12) for  $r = 2$  and (3.17), we obtain (3).

Since  $\frac{\partial d_M}{\partial z_1 \partial \bar{z}_1} = \frac{1}{2}$  (see (3.8) and (3.15)), property (1) follows immediately from (3.6).  $\square$

The following lemma is essential in the proof of Theorem 4.2, where regular Stein neighborhoods are constructed.

**Lemma 3.3.** *Let  $A$  be a real diagonalizable  $n \times n$  matrix, and let  $d_M$  and  $d_N$  respectively be the squared Euclidean distance functions to  $M = (A + iI)\mathbb{R}^n$  and  $N = \mathbb{R}^n$ . If  $A$  satisfy one of the conditions*

- (1)  $A = \text{diag}(a, \dots, a)$ , where  $|a| \leq \frac{1}{\sqrt{15}}$  (respectively  $|a| \leq \frac{5}{\sqrt{11}}$ ) for  $n \geq 2$  (respectively  $n = 2$ ),
- (2)  $A = \text{diag}(a_1, a_2)$ , where  $|a_1|, |a_2| \leq \frac{1}{\sqrt{15}}$ , ( $n=2$ ),
- (3)  $A = \begin{bmatrix} c & -b \\ b & c \end{bmatrix}$ , where  $|c| \leq \frac{1}{8}$ ,  $|b| \leq \frac{1}{16}$ , ( $n=2$ ),

then there exists a homogeneous polynomial  $P \in \mathbb{R}[u, v]$  such that the function

$$\rho = P(d_M, d_N).$$

is strictly plurisubharmonic everywhere except at the origin, and such that

$$(3.18) \quad \begin{aligned} \{P = 0\} \cap \mathbb{R}_+^2 &= (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+), \\ \{\nabla P = 0\} \cap \mathbb{R}_+^2 &= \{0\}, \quad \frac{\partial P}{\partial u}(u, 0) = 0 = \frac{\partial P}{\partial v}(0, v), \quad u, v \in \mathbb{R}_+. \end{aligned}$$

*Proof.* Recall that  $d_N(x, y) = \sum_{j=1}^n y_j^2$  and  $\frac{\partial d_N}{\partial z_j}(x, y) = -iy_j$ .

If  $A = \text{diag}(a_1, \dots, a_n)$  is the  $n \times n$  real diagonal matrix, then  $d_M(x, y) = \sum_{j=1}^n \frac{x_j - a_j y_j}{1 + a_j^2}$  (see Lemma 2.1), and for any  $j \in \{1, \dots, n\}$  we further get  $\frac{\partial d_M}{\partial z_j} = \frac{(1 + ia_j)(x_j - a_j y_j)}{1 + a_j^2}$ . An easy and straightforward calculation then yields

$$\text{Re} \langle (\frac{\partial d_M}{\partial z})_r, (\frac{\partial d_N}{\partial z})_r \rangle = - \sum_{j=1}^r \frac{a_j(x_j - a_j y_j) y_j}{1 + a_j^2}, \quad r \in \{1, \dots, n\},$$

and using the Cauchy-Schwarz inequality, we obtain that

$$(3.19) \quad \left| \text{Re} \langle (\frac{\partial d_M}{\partial z})_r, (\frac{\partial d_N}{\partial z})_r \rangle \right| \leq \vartheta \left( \sum_{j=1}^r \frac{(x_j - a_j y_j)^2}{1 + a_j^2} \sum_{j=1}^r y_j^2 \right)^{\frac{1}{2}} \leq (d_{M,r} d_{N,r})^{\frac{1}{2}}$$

where we denoted  $d_{M,r} = \sum_{j=1}^r \frac{(x_j - a_j y_j)^2}{1 + a_j^2}$ ,  $d_{N,r} = \sum_{j=1}^r y_j^2$  and  $\vartheta = \max_{1 \leq j \leq r} \frac{|a_j|}{1 + a_j^2}$ . We also compute  $Z_{jk}(A)$  in (3.4):

$$\begin{aligned} Z_{jk}(A) &= \sum_{m=j,k} |\frac{\partial d_M}{\partial z_m}|^2 \sum_{m=j,k} |\frac{\partial d_N}{\partial z_m}|^2 - \left| \sum_{m=j,k} \frac{\partial d_M}{\partial z_m} \frac{\partial d_N}{\partial \bar{z}_m} \right|^2 \\ &= \frac{(x_j - a_j y_j)^2 y_j^2}{1 + a_j^2} - \frac{2(1 + a_j a_k)(x_j - a_j y_j)(x_k - a_k y_k) y_j y_k}{(1 + a_j^2)(1 + a_k^2)} + \frac{(x_k - a_k y_k)^2 y_k^2}{1 + a_k^2} \end{aligned}$$

Now, let us consider the case  $A = \text{diag}(a, \dots, a)$  for  $a \in \mathbb{R}$ . It then follows that

$$\begin{aligned} \sum_{1 \leq j < k \leq r} Z_{jk}(A) &= \sum_{1 \leq j < k \leq r} \left( \frac{(x_j - a y_j)^2 y_j^2}{1 + a^2} - \frac{2(1 + a^2)(x_j - a y_j)(x_k - a y_k) y_j y_k}{(1 + a^2)(1 + a^2)} + \frac{(x_k - a y_k)^2 y_k^2}{1 + a^2} \right) \\ &= \sum_{1 \leq j < k \leq r} \left( \frac{(x_j - a y_j) y_j}{\sqrt{1 + a^2}} - \frac{(x_k - a y_k) y_k}{\sqrt{1 + a^2}} \right)^2 \\ &= \sum_{j=1}^r \frac{(x_j - a y_j)^2}{1 + a^2} \sum_{j=1}^r y_j^2 - \left( \sum_{j=1}^r \frac{(x_j - a y_j) y_j}{\sqrt{1 + a^2}} \right)^2, \end{aligned}$$

where the last equality is obtained by Lagrange's identity. Thus

$$(3.20) \quad 0 \leq \sum_{1 \leq j < k \leq r} Z_{jk}(A) \leq d_{M,r} d_{N,r}, \quad 2 \leq r \leq n.$$

Using Lemma 3.2 (2) for  $P(u, v) = u^2 v + v^2 u$  with  $\rho = P(d_M, d_N)$ , and combining with the estimates (3.19), (3.20), we obtain for  $r \geq 2$  that

$$(3.21) \quad \begin{aligned} \Delta^{2-r} \det(H_r^C(\rho)) &\geq \Delta^2 + \Delta(2d_N d_{M,r} + 2d_M d_{N,r} - 4\vartheta(d_M + d_N)(d_{M,r} d_{N,r})^{\frac{1}{2}}) \\ &\quad - 4(d_M^2 + d_M d_N + d_N^2) d_{M,r} d_{N,r}, \end{aligned}$$

where  $\Delta = \frac{1}{2}(d_M^2 + 4d_M d_N + d_N^2)$  and  $\det(H^{\mathbb{R}}(P)_{(d_M, d_N)}) = -4(d_M^2 + d_M d_N + d_N^2)$ . For any  $r \in \{1, \dots, n\}$  we denote the expression on the right-hand side of the above

inequality by  $\Psi_{r,\vartheta}$ . Observe that for  $\vartheta = \frac{1}{4}$  and regrouping the terms of  $\Psi_{r,\frac{1}{4}}$ , we get

$$\begin{aligned}
 \Psi_{r,\frac{1}{4}} = & d_M d_N (2d_{M,r} d_N + 2d_{N,r} d_M - 4d_{M,r} d_{N,r}) \\
 & + (d_M^2 + d_N^2)(2d_M d_N + d_M d_{N,r} + d_{M,r} d_N - 4d_{M,r} d_{N,r}) \\
 (3.22) \quad & + d_M^2 d_N (2d_{N,r} - \frac{5}{2}(d_{N,r} d_{M,r})^{\frac{1}{2}} + \frac{25}{32} d_M) \\
 & + d_N^2 d_M (2d_{M,r} - \frac{5}{2}(d_{N,r} d_{M,r})^{\frac{1}{2}} + \frac{25}{32} d_N) \\
 & + \frac{1}{8}(d_M^3 + d_N^3)(d_M + 4(d_{M,r} d_{N,r})^{\frac{1}{2}} + 4d_N) \\
 & + \frac{1}{8} d_M^4 - \frac{41}{32} d_M^3 d_N + \frac{9}{2} d_M^2 d_N^2 - \frac{41}{32} d_M d_N^3 + \frac{1}{8} d_N^4.
 \end{aligned}$$

Since we have  $d_M \geq d_{M,r}$  and  $d_N \geq d_{N,r}$  for all  $r$ , the first five terms in the above sum are non-negative. The term is a symmetric homogeneous polynomial in  $d_M$  and  $d_N$ . It is clearly positive on  $(M \cup N) \setminus \{0\}$ , while on  $\mathbb{C}^n \setminus (M \cup N)$  we can factor out  $d_M^2 d_N^2$  and set  $W = \frac{d_M}{d_N} + \frac{d_N}{d_M}$ , and after regrouping the terms we obtain  $d_M^2 d_N^2 (\frac{1}{8} W^2 - \frac{41}{32} W + \frac{17}{4})$ , which is also positive. It follows that  $\Psi_{r,\frac{1}{4}}$  vanishes exactly at the origin and is positive elsewhere. Moreover, this holds for any  $\Psi_{r,\vartheta}$ ,  $|\vartheta| \leq \frac{1}{4}$ . Remember that  $A = \text{diag}(a, \dots, a)$  and therefore  $\vartheta = \frac{|a|}{\sqrt{1+a^2}}$ . Since  $f(x) = \frac{x}{\sqrt{1+x^2}}$  is an increasing function,  $|a| \leq \frac{1}{\sqrt{15}}$  implies that  $\det(H_r^{\mathbb{C}}(\rho)) \geq \Psi_{r,\frac{1}{4}} \geq 0$  for all  $r \in \{2, \dots, n\}$  with equalities precisely at the origin.

Next, applying (3.3) and using (3.19), we get

$$(3.23) \quad \det(H_1^{\mathbb{C}}(\rho)) \geq \Delta + (2d_N d_{M,1} + 2d_M d_{N,1} - 4\vartheta(d_M + d_N)(d_{M,1} d_{N,1})^{\frac{1}{2}}).$$

Comparing the above inequality to (3.21) we see that  $\Delta \det(H_1^{\mathbb{C}}(\rho)) \geq \Psi_{1,\vartheta}$ . It thus follows that  $\det(H_1^{\mathbb{C}}(\rho))$  for  $|a| \leq \frac{1}{\sqrt{15}}$  (hence  $\vartheta \leq \frac{1}{4}$ ) vanishes at the origin and is positive elsewhere. This concludes the proof that  $\rho = d_M^2 d_N + d_M d_N^2$  is strictly plurisubharmonic everywhere, except at the origin, in the case when  $M = M(A)$  with  $A = \text{diag}(a, \dots, a)$ ,  $|a| \leq \frac{1}{\sqrt{15}}$ .

We proceed with the case  $A = \text{diag}(a_1, a_2)$ . Here we have (see Lemma 3.2 (2)):

$$(3.24) \quad Z_{12}(A) = d_M d_N - \left| \sum_{j=1,2} \frac{\partial d_M}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_j} \right|^2 \leq d_M d_N.$$

Using Lemma 3.2 (2),(2) for  $P(u, v) = u^2 v + v^2 u$  ( $\rho = P(d_M, d_N)$ ), again, together with the estimates (3.19) and (3.24), we obtain exactly the same estimates as in (3.21) for  $r = n = 2$  and (3.23). By the same argument as in the case  $A = \text{diag}(a, \dots, a)$ , it then follows that for  $\vartheta = \max_{1 \leq j \leq 2} \frac{|a_j|}{\sqrt{1+a_j^2}} \leq \frac{1}{4}$  ( $\max_{1 \leq j \leq 2} |a_j| \leq \frac{1}{\sqrt{15}}$ ) we obtain  $\det(H_r^{\mathbb{C}}(\rho)) \geq 0$ ,  $r \in \{1, \dots, n\}$ , with equality precisely at the origin.

If  $n = 2$  and  $A$  has complex eigenvalues, remember that by (2.4) we have  $d_M(x_1, y_1, x_2, y) = \frac{(x_1 - cy_1 + by_2)^2}{1+b^2+c^2} + \frac{(x_2 - cy_2 - by_1)^2}{1+b^2+c^2}$  and the holomorphic derivatives of  $d_M$  are of the form (3.16). By regrouping the terms we easily obtain

$$(3.25) \quad \text{Re}(\frac{\partial d_M}{\partial z_1} \frac{\partial d_N}{\partial \bar{z}_1}) = \frac{1}{1+b^2+c^2} y_1 (c(x_1 - cy_1 + by_2) + b(x_2 - cy_2 - by_1))$$

and

$$(3.26) \quad \operatorname{Re}\left(\sum_{j=1,2} \frac{\partial d_M}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_j}\right) = \frac{1}{1+b^2+c^2}(-by_2(x_1 - cy_1 + by_2) + by_1(x_2 - cy_2 - by_1)) \\ + \frac{1}{1+b^2+c^2}(cy_1(x_1 - cy_1 + by_2)cy_2(x_2 - cy_2 - by_1)).$$

Applying Cauchy-Schwarz inequality respectively to (3.25) and to each term of the above sum (3.26), and by possibly using triangle inequality, we get

$$(3.27) \quad \left|\operatorname{Re}\left(\frac{\partial d_M}{\partial z_1} \frac{\partial d_N}{\partial \bar{z}_1}\right)\right| \leq \frac{\sqrt{b^2+c^2}}{\sqrt{1+b^2+c^2}}(d_M d_N)^{\frac{1}{2}}$$

and

$$(3.28) \quad \left|\operatorname{Re}\left(\sum_{j=1,2} \frac{\partial d_M}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_j}\right)\right| \leq \frac{|b|+|c|}{\sqrt{1+b^2+c^2}}(d_M d_N)^{\frac{1}{2}}$$

We apply Lemma 3.2 (1), (3), once more for  $\rho = P(d_M, d_N)$ , where  $P(u, v) = u^2v + v^2u$ . Using (3.27) and  $\frac{\sqrt{b^2+c^2}}{\sqrt{1+b^2+c^2}} \leq \frac{1}{4}$  ( $|b| \leq \frac{1}{16}$ ,  $|c| \leq \frac{1}{8}$ ), and after regrouping the terms, we obtain

$$\det(H_1^C(\rho)) \geq \Delta + (2d_N|\frac{\partial d_M}{\partial z_1}|^2 + 2d_M|\frac{\partial d_N}{\partial z_1}|^2 - 4\frac{\sqrt{b^2+c^2}}{\sqrt{1+b^2+c^2}}(d_M + d_N)(d_M d_N)^{\frac{1}{2}}) \\ \geq \frac{1}{2}(d_M^2 + 4d_M d_N + d_N^2) - (d_M + d_N)(d_M d_N)^{\frac{1}{2}} \\ \geq \frac{1}{2}(d_M + d_N)(\sqrt{d_M} - \sqrt{d_N})^2 + d_M d_N.$$

It follows that  $\det(H_1^C(\rho))$  vanishes at the origin and is positive elsewhere. Next, (3.5) and (3.28) yield

$$(3.29) \quad \det(H_2^C(\rho)) \geq \Delta^2 + \Delta(2d_N d_M + 2d_M d_N - 4\frac{|b|+|c|}{\sqrt{1+b^2+c^2}}(d_M + d_N)(d_M d_N)^{\frac{1}{2}}) \\ - 4(d_M^2 + d_M d_N + d_N^2)d_M d_N - (\frac{b}{1+b^2+c^2}(2d_M d_N + d_N^2))^2 \\ - \frac{4b^2}{1+b^2+c^2}(2d_M^2 d_N^2 + d_N^3 d_M),$$

where  $\Delta = \frac{1}{2}(d_M^2 + 4d_M d_N + d_N^2)$  is as before. Using the estimates  $1 + b^2 + c^2 \geq 1$  and  $|c| \leq \frac{1}{8}$ ,  $|b| \leq \frac{1}{16}$  respectively, and regrouping the terms, we further get

$$(3.30) \quad \det(H_2^C(\rho)) \geq \frac{1}{4}d_M^4 + \frac{17}{2}d_M^2 d_N^2 + \frac{1}{4}d_M^4 - b^2(2d_M d_N + d_N^2)^2 - b^2 d_N^2(8d_M^2 + 4d_N d_M) \\ - 2(|b| + |c|)(d_M^{\frac{7}{2}} + 5d_M^2 d_N + 5d_M d_N^2 + d_N^3)(d_M d_N)^{\frac{1}{2}} \\ \geq (\frac{1}{16}d_M^4 + \frac{136}{512}d_M^3 d_N) + d_M^2 d_N(2d_N - \frac{15}{8}(d_N d_M)^{\frac{1}{2}} + \frac{125}{521}d_M) \\ + d_M d_N^2(2d_M - \frac{15}{8}(d_N d_M)^{\frac{1}{2}} + \frac{125}{521}d_N) \\ + \frac{1}{8}d_M^3(d_M - 3(d_M d_N)^{\frac{1}{2}} - \frac{9}{4}d_N) + \frac{1}{8}d_N^3(d_N - 3(d_M d_N)^{\frac{1}{2}} - \frac{9}{4}d_M) \\ + (\frac{1}{16}d_M^4 - \frac{505}{512}d_M^3 d_N + \frac{135}{32}d_M^2 d_N^2 - \frac{505}{512}d_M d_N^3 + \frac{1}{16}d_N^4).$$

The first five terms are clearly non-negative, while the last one is positive everywhere, except at the origin. Indeed, on  $\mathbb{C}^2 \setminus (M \cup N)$  the last term can be seen as  $d_M^2 d_N^2(W^2 - \frac{505}{512}W + \frac{132}{32})$ ,  $W = \frac{d_M}{d_N} + \frac{d_N}{d_M}$ . It follows that  $\rho = d_M^2 d_N + d_M d_N^2$  is strictly plurisubharmonic everywhere, except at the origin.

Finally, let  $A = \operatorname{diag}(a, a)$ ,  $a \in \mathbb{R}$ . We have  $\operatorname{Re}\left(\frac{\partial d_M}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_j}\right) = -\sum_{j=1}^2 \frac{a(x_j - ay_j)y_j}{1+a^2}$  and  $\left|\sum_{j=1,2} \frac{\partial d_M}{\partial z_j} \frac{\partial d_N}{\partial \bar{z}_j}\right|^2 = \frac{1}{1+a^2}\left(\sum_{j=1}^2 (x_j - ay_j)y_j\right)^2$ . Setting  $\vartheta = \frac{a}{\sqrt{1+a^2}}$  and  $Z =$

$\frac{1}{\sqrt{1+a^2}}(\sum_{j=1}^2(x_j - ay_j)y_j)$ , Lemma 3.2 (2) for a polynomial  $P$  and  $\rho = P(d_M, d_N)$  then yields

$$(3.31) \quad \det(H_2^{\mathbb{C}}(\rho)) = \Delta\left(\frac{\partial^2 P}{\partial u^2}(d_M, d_N)d_M + \frac{\partial^2 P}{\partial v^2}(d_M, d_N)d_N - 2\frac{\partial^2 P}{\partial u \partial v}(d_M, d_N)\vartheta Z\right) + \Delta^2 + \det(H^{\mathbb{R}}(P)_{(d_M, d_N)})(d_M d_N - Z^2).$$

By choosing  $P(u, v) = u^3 v + 4u^2 v^2 + uv^3$ , we have  $\det(H^{\mathbb{R}}(P)_{(d_M, d_N)}) = -(9d_M^4 - 48d_M^3 d_N + 174d_M^2 d_N^2 + 48d_M d_N^3 + 9d_N^4)$  and  $\Delta = \frac{1}{2}(d_M^3 + 13d_M^2 d_N + 13d_M d_N^2 + d_N^3)$ , and we further obtain

$$(3.32) \quad \det(H_2^{\mathbb{C}}(\rho)) = R(d_M, d_N)Z^2 + \vartheta S(d_M, d_N)Z + T(d_M, d_N),$$

where

$$\begin{aligned} R(u, v) &= 3(3u^4 + 20u^3 v + 94u^2 v^2 + 20uv^3 + 3v^2), \\ S(u, v) &= -\vartheta(3u^5 + 59u^4 v + 302u^3 v^2 + 302u^2 v^3 + 59uv^4 + 3v^5), \\ T(u, v) &= \frac{1}{4}(u^6 + 22u^5 v + 403u^4 v^2 + 44u^3 v^3 + 403u^2 v^4 + 22uv^5 + v^6). \end{aligned}$$

The discriminant of the ekspression (3.32) with respect to  $Z$  is for  $|\vartheta| = \frac{5}{6}$  equal to

$$\begin{aligned} &-\frac{3}{16}(21u^{10} + 314u^9 v + 12009u^8 v^2 + 52136u^7 v^3 + 265042u^6 v^4 - 241956u^5 v^5 \\ &+ 265042u^4 v^6 + 52136u^3 v^7 + 12009u^2 v^8 + 314uv^9 + 21v^{10}), \end{aligned}$$

which is negative everywhere, except at  $(u, v) = (0, 0)$ . It follows that for  $|a| \leq \frac{5}{\sqrt{11}}$  (and hence  $|\vartheta| \leq \frac{5}{6}$ ) we have  $\det(H_2^{\mathbb{C}}(\rho)) \geq 0$  with equality precisely at the origin.

From (3.19) in (3.3) we obtain

$$(3.33) \quad \begin{aligned} \det(H_1^{\mathbb{C}}(\rho)) &= (2d_N(6d_M + 5d_N)d_{M,1} + 2d_M(6d_N + 5d_M)d_{N,1}) \\ &+ \frac{1}{2}(d_M^3 + 13d_M^2 d_N + 13d_M d_N^2 + d_N^3) \\ &- \vartheta(3d_M^2 + 16d_M d_N + 3d_N^2)(d_{M,1}d_{N,1})^{\frac{1}{2}}. \end{aligned}$$

Regrouping the terms for  $|\vartheta| \leq \frac{5}{6}$  ( $|a| \leq \frac{5}{\sqrt{11}}$ ), we further get

$$\begin{aligned} \det(H_1^{\mathbb{C}}(\rho)) &\geq d_M^2\left(\frac{1}{2}d_M - \frac{9}{2}(d_{M,1}d_{N,1})^{\frac{1}{2}} + 10d_{N,1} + \frac{9}{8}d_N\right) \\ &+ d_N^2\left(\frac{1}{2}d_N - \frac{9}{2}(d_{N,1}d_{M,1})^{\frac{1}{2}} + 10d_{M,1} + \frac{9}{8}d_M\right) \\ &+ 6d_N d_M(d_{M,1} - 2(d_{M,1}d_{N,1})^{\frac{1}{2}} + d_{N,1}) \\ &+ 6d_N d_M(d_M - 2(d_{M,1}d_{N,1})^{\frac{1}{2}} + d_N) + \frac{3}{4}d_M d_N(d_M + d_N). \end{aligned}$$

Since  $d_M \geq d_{M,1}$  and  $d_N \geq d_{N,1}$ , all the terms in the right hand-side of the equality are non-negative, and in addition the last one vanishes precisely at the origin. This proves that  $\rho = d_M^3 + d_M^2 d_N + d_M d_N^2 + d_N^3$  for  $|a| \leq \frac{5}{\sqrt{11}}$  ( $M = M(A)$  with  $A = \text{diag}(a, a)$ ) is strictly plurisubharmonic everywhere, except at the origin.

To concludes the proof of the lemma we observe that in both cases, if  $P$  is either  $P(u, v) = u^3 v + 4u^2 v^2 + uv^3$  or  $P(u, v) = u^3 v + 4u^2 v^2 + uv^3$ , the property (3.18) is satisfied.  $\square$

*Remark 3.4.* The estimates on the entries of  $A$  in the lemma are certainly not optimal and might be improved, while on the other hand it is not clear at the moment

of this writing, how to obtain a significantly better estimates. The computations quickly get very lengthy if we increase the degree of the polynomial  $P$ .

#### 4. REGULAR STEIN NEIGHBORHOODS

A system of open Stein neighborhoods  $\{\Omega_\epsilon\}_{\epsilon \in (0,1)}$  of a set  $S$  in a complex manifold  $X$  is called a *regular*, if for every  $\epsilon \in (0,1)$  we have

- (1)  $\Omega_\epsilon = \cup_{t < \epsilon} \Omega_t$ ,  $\overline{\Omega}_\epsilon = \cap_{t > \epsilon} \Omega_t$ ,
- (2)  $S = \cap_{\epsilon \in (0,1)} \Omega_\epsilon$  is a strong deformation retract of every  $\Omega_\epsilon$  with  $\epsilon \in (0,1)$ .

For instance, one way to construct such a system of neighborhoods is to find a non-negative function  $\rho$ , which is strictly plurisubharmonic in some neighborhood of  $S$ , and such that  $S = \{\rho = 0\} = \{\nabla \rho = 0\}$ . Observe that in this case the sublevel sets  $\Omega_\epsilon = \{\rho < \epsilon\}$  for  $\epsilon$  small enough are Stein, and the flow of the negative gradient vector field  $-\nabla \rho$  gives us the strong deformation retraction of  $\Omega_\epsilon$  to  $S$ . Note that slightly weaker conditions concerning plurisubharmonicity of  $\rho$  can work as well (see e.g. Theorem 4.2 or [14, Theorem 4.1]).

For the sake of completeness we also recall the following fact about homogeneous polynomials [14, Lemma 3.2], which will be used later on.

**Lemma 4.1.** *Let  $Q, R \in \mathbb{R}[x_1, x_2, \dots, x_m]$  be real homogeneous polynomials in  $m$  variables and of even degree  $s$ . Assume further that  $Q$  is vanishing at the origin and is positive elsewhere. Then for any sufficiently small constant  $\epsilon_0 > 0$ , it follows that  $Q \geq \epsilon_0 \cdot |R|$ , with equality precisely at the origin.*

We are now ready to prove the main result.

**Theorem 4.2.** *Let  $A$  be a real  $n \times n$  matrix such that  $A - iI$  is invertible. Further, let  $M(A) = (A + iI)\mathbb{R}^2$ . Then the union  $M(A) \cup \mathbb{R}^n$  has a regular system of strongly pseudoconvex Stein neighborhoods if any of the following properties are satisfied:*

- (1) *The real parts of the eigenvalues of  $A$  are sufficiently close to a real constant  $a$  with  $|a| \leq \frac{1}{\sqrt{15}}$  ( $|a| \leq \frac{5}{\sqrt{11}}$  if  $n = 2$ ), while the imaginary parts are sufficiently close to zero.*
- (2)  *$n = 2$  and moduli of the real parts of the eigenvalues of  $A$  are  $\leq \frac{1}{\sqrt{15}}$ , while the imaginary parts are sufficiently close to zero.*
- (3)  *$n = 2$  and the moduli of the real (respectively imaginary) parts of the eigenvalues of  $A$  are  $\leq \frac{1}{8}$  (respectively  $\leq \frac{1}{16}$ ).*

*Moreover, away from the origin the neighborhoods coincide with sublevel sets of the squared Euclidean distance functions to  $M$  and  $N$  respectively.*

It is clear that non-singular linear transformations map a regular system of Stein neighborhoods into a regular system of Stein neighborhoods. According to the note in Section 2, the general case of the union of two totally real subspaces  $M, N$  of maximal dimension, intersecting at the origin, thus reduces to the situation described in the Theorem 4.2, i.e.  $N = \mathbb{R}^2$  and  $M = (A + iI)\mathbb{R}^n$ , where  $i$  is not the eigenvalue of the real  $n \times n$  matrix  $A$ .

*Proof of the Theorem 4.2.* Our goal is to construct the function  $\rho$ , which is strictly plurisubharmonic everywhere, except maybe at the origin, and such that it satisfies

the condition  $M(A) \cup \mathbb{R}^n = \{\rho = 0\} = \{\nabla \rho = 0\}$ . Clearly, since any real non-singular matrix  $R$  maps  $M(A) = (A + iI)\mathbb{R}^n$  onto  $M(VAV^{-1}) = (VAV^{-1} + iI)\mathbb{R}^n$ , we deduce that  $\sigma = \rho \circ V^{-1}$  with respect to  $M(VAV^{-1})$  inherits all the above properties of  $\rho$ . It is therefore sufficient to consider the case when  $A$  is in the Jordan canonical form (2.3), where the parameter  $\delta$  can be chosen arbitrarily.

If  $A$  satisfies property (3) then Lemma 3.3(3) immediately implies the existence of the function  $\rho$  with the properties listed above.

Observe that for any real  $n \times n$  matrix  $B$  and any homogeneous polynomial  $P$  of degree  $k \geq 2$  in two variables, it follows that  $\det(H_r^{\mathbb{C}}(P(d_{M(B)}, d_N)))$ ,  $r \in \{1, \dots, n\}$ , is a homogeneous polynomial of degree  $(2k - 2)r$  in  $x, y$ .

Next, let  $A = A_\delta$  be of the form (2.3), where  $\delta$  is to be chosen later. By Lemma 2.1 we have  $d_{M(A)} = d_{M(A_0)} + q_\delta$ , where  $q_\delta$  is a homogeneous polynomial of degree 2 in variables  $x_1, \dots, x_n, y_1, \dots, y_n$  and such that its coefficients are rational functions in  $\delta$  and they have no pole at  $\delta = 0$ . For a homogeneous polynomial  $P$  (to be chosen) of degree  $k \geq 2$  in two variables we obtain

$$(4.1) \quad \det(H_r^{\mathbb{C}}(P(d_{M(A_\delta)}, d_N))) = \det(H_r^{\mathbb{C}}(P(d_{M(A_0)}, d_N))) + Q_\delta, \quad r \in \{1, \dots, n\},$$

where  $Q_\delta$  is a homogeneous polynomial of degree  $(2k - 2)r$  in  $x, y$ , and in addition its coefficients are rational functions in  $\delta$  and without a pole at  $\delta = 0$ .

Let further

$$A_0 = \text{diag}(D_1, \dots, D_\beta, d_1, \dots, d_\gamma),$$

$$D_j = C_j + \epsilon_j I_2, \quad j \in \{1, \dots, \beta\}, \quad d_k = a_k + \epsilon_{k+\beta}, \quad k \in \{1, \dots, \gamma\},$$

where  $I_2$  is the  $2 \times 2$  identity-matrix,  $C_j = \begin{bmatrix} c_j & -b_j \\ b_j & c_j \end{bmatrix}$  is a real matrix and  $a_1, \dots, a_\gamma, \epsilon_1, \dots, \epsilon_{\beta+\gamma} \in \mathbb{R}$ . Setting  $B_0 = \text{diag}(C_1, C_2, \dots, C_\beta, a_1, \dots, a_\gamma)$ , we observe that  $d_{M(A_0)} = d_{M(B_0)} + s_\epsilon$  and

$$(4.2) \quad \det(H_r^{\mathbb{C}}(P(d_{M(A_0)}, d_N))) = \det(H_r^{\mathbb{C}}(P(d_{M(B_0)}, d_N))) + S_\epsilon, \quad r \in \{1, \dots, n\},$$

where  $s_\epsilon$  and  $S_\epsilon$  respectively are homogeneous polynomials of degrees 2 and  $2kr$  in  $x, y$ , and such that their coefficients are polynomials in variables  $\epsilon_1, \dots, \epsilon_{\beta+\gamma}$  without constant term. For any  $j \in \{1, \dots, \beta\}$  we have

$$\begin{aligned} & \frac{(x_{2j-1} - c_j y_{2j-1} + b_j y_{2j})^2 + (x_{2j} - c_j y_{2j} - b_j y_{2j-1})^2}{1 + c_j^2 + b_j^2} = \\ & = \frac{(x_{2j-1} - c_j y_{2j-1})^2 + (x_{2j} - c_j y_{2j})^2}{1 + c_j^2 + b_j^2} + b_j^2 \frac{(x_{2j-1} - c_j y_{2j-1})^2 + (x_{2j} - c_j y_{2j})^2}{(1 + c_j^2 + b_j^2)(1 + c_j^2)} \\ & + b_j \frac{(2(x_{2j-1} - c_j y_{2j-1}) + b_j y_{2j})y_{2j} - (2(x_{2j} - c_j y_{2j}) - b_j y_{2j-1})y_{2j-1}}{(1 + c_j^2 + b_j^2)}. \end{aligned}$$

Using (2.4) it thus follows

$$d_{M(A_0)}(x, y) = d_{M(\Lambda)}(x, y) + \sum_{j=1}^{\beta} t_j(x, y),$$

where  $\Lambda = \text{diag}(c_1, c_1, \dots, c_\beta, c_\beta, a_1, \dots, a_\gamma)$ , and for every  $j$  the polynomial  $t_j$  is homogeneous polynomial of degree 2 in variables  $x_{2j-1}, x_{2j}, y_{2j-1}, y_{2j}$ , and such



that its coefficients are rational functions in  $b_j$  and they have a zero at  $b_j = 0$ . It further implies that

$$(4.3) \quad \det(H_r^{\mathbb{C}}(P(d_{M(A_0)}, d_N))) = \det(H_r^{\mathbb{C}}(P(d_{M(\Lambda)}, d_N))) + T_b, \quad r \in \{1, \dots, n\},$$

where  $T_b$  is a homogeneous polynomial of degree  $(2k-2)r$  in  $x, y$ . In addition, the coefficients of  $s_b$  are rational functions in  $b_1, \dots, b_\beta$ , and such that they vanish for  $b_1 = \dots = b_\beta = 0$ .

Observe that  $A_\delta$  satisfies the condition (1) (respectively (2)) of Lemma 4.2 precisely when  $\Lambda$  satisfies the condition (1) (respectively (2)) of Lemma 3.3, provided that constants  $b_1, \dots, b_\beta, \epsilon_1, \dots, \epsilon_{\beta+\gamma}, \delta$  are small enough. Furthermore, if  $\Lambda$  satisfies any of the conditions (1) or (2) in Lemma 3.3, then there exists a homogeneous polynomial  $P$ , and such that  $\rho = P(d_{M(\Lambda)}, d_N)$  is a polynomial in  $x, y$ , which is strictly plurisubharmonic everywhere except at the origin, and such that (3.18) is satisfied. We now use Lemma 4.1 to see that for sufficiently small constants  $b_1, \dots, b_\beta, \epsilon_1, \epsilon_{\beta+\gamma}, \delta$ , and using (4.3), (4.2), (4.1), respectively,  $\det(H_r^{\mathbb{C}}(P(d_{M(A_\delta)}, d_N)))$  vanishes at the origin and is positive everywhere else. Since  $\nabla \rho = \frac{\partial P}{\partial u}(d_{M(A_\delta)}, d_N) \nabla d_{M(A_\delta)} + \frac{\partial P}{\partial v}(d_{M(A_\delta)}, d_N) \nabla d_N$ , it follows from (3.18) that  $M \cup N = \{\rho = 0\} = \{\nabla \rho = 0\}$ .

Finally, mutatis mutandis, the proof given in [14, Theorem 4.1] now applies to glue  $\rho$  away from the origin with the squared distance functions. We choose open balls  $B_R$  and  $B_{2R}$  respectively, centered at 0 and with radii  $R$  and  $2R$ , and observe that for any sufficiently small  $\epsilon > 0$  the sets

$$T_{\epsilon, M} = \{z \in \mathbb{C}^n \setminus \overline{B_R} : d_M(z) < \epsilon\}, \quad T_{\epsilon, N} = \{z \in \mathbb{C}^n \setminus \overline{B_R} : d_N(z) < \epsilon\}$$

are disjoint. Next, we set  $T_\epsilon = T_{\epsilon, M} \cup T_{\epsilon, N}$  and define:

$$\rho_0(z) = \theta(z)\rho(z) + (1 - \theta(z))d_M|_{T_{\epsilon, M}}(z) + (1 - \theta(z))d_N|_{T_{\epsilon, N}}(z), \quad z \in B_{2R} \cup T_\epsilon.$$

Here  $\theta(z) = \chi(\sum_{j=1}^n |z_j|^2)$ , where  $\chi$  is a suitable cut-off function with  $\chi(t) = 1$  for  $t \leq R$  and  $\chi(t) = 0$  for  $t \geq 2R$ .

It is clear that  $\{\rho_0 = 0\} = M \cup N$ . On  $(B_{2R} \setminus \overline{B_R}) \setminus (M \cup N)$ , but close to  $M \cup N$ , we have  $\nabla \theta$  near to tangent directions to  $M \cup N$ , and  $\nabla d_M$  or  $\nabla d_N$  respectively are near to normal directions to  $M$  and  $N$ . Hence, after possibly choosing  $\epsilon$  smaller we obtain  $\{\nabla \rho_0 = 0\} = M \cup N$ . The flow of the negative gradient vector field  $-\nabla \rho_0$  then yields a deformation retraction of  $\Omega_\epsilon = \{\rho_0 < \epsilon\}$  onto  $M \cup N$ .

Since  $\rho, d_M, d_N$  along with their gradients all vanish on  $M \cup N$ , it follows that for  $z \in M \cup N$  and any  $\xi \in T_z(\mathbb{C}^n)$  we have

$$\mathcal{L}_{(z)}(\rho_0; \xi) = \theta(z)\mathcal{L}_{(z)}(\rho; \xi) + (1 - \theta(z))\mathcal{L}_{(z)}(d_M|_{T_{\epsilon, M}}; \xi) + (1 - \theta(z))\mathcal{L}_{(z)}(d_N|_{T_{\epsilon, N}}; \xi).$$

The Levi form of  $\rho_0$  is thus positive on  $\overline{\Omega}_\epsilon \setminus \{0\}$ , provided that  $\epsilon$  is chosen small enough. Further, since the restrictions of plurisubharmonic functions to analytic sets are plurisubharmonic and satisfy the maximum principle (see [8]), there cannot be any compact analytic subset of positive dimension in  $\mathbb{C}^n$ . By a result of Grauert (see [7, Proposition 5])  $\Omega_\epsilon$  is then Stein. This completes the proof.  $\square$

Lemma 3.3 can be also applied to prove the existence of regular neighborhoods of certain smooth totally real immersions of a real  $n$ -manifolds into a complex  $n$ -manifold; for results on closed real surfaces immersed into complex surface see [5, Theorem 2.2], [13, Theorem 2], [14, Proposition 4.3]).

**Proposition 4.3.** *Let  $\pi: Z \rightarrow X$  be an smooth totally real immersion of a closed real  $n$ -manifold into a complex  $n$ -manifold  $X$ , and such that  $\pi$  has only transverse double points (no multiple points)  $q_1, \dots, q_k \in \pi(Z)$  with  $\pi^{-1}(q_j) = \{t_j, u_j\}$ . For any  $j \in \{1, \dots, s\}$ , let the images under tangent map of the tangent spaces of  $Z$  at points  $t_j$  and  $u_j$ , respectively, define a union of two totally real subspaces in  $T_{q_j}X \approx \mathbb{C}^n$ , which is holomorphically-equivalent to  $(A_j + iI)\mathbb{R}^n \cup \mathbb{R}^n \subset \mathbb{C}^n$ , where  $A_j$  is a real  $n \times n$  matrix with  $A_j - iI$  invertible. If the entries of  $A_j$  for all  $j \in \{1, \dots, k\}$  satisfy any of the conditions (1), (2) or (3) in Lemma 3.3, then  $\tilde{Z} = \pi(Z)$  has a regular Stein neighborhood basis.*

*Proof.* For any double point  $q_j$  there exists local holomorphic coordinates  $\psi_j: U_j \rightarrow V_j \subset \mathbb{C}^n$ , such that  $\psi_j(q_j) = 0$  and such that  $\psi_j(\tilde{Z}) = S_j \cup T_j$ , where  $S_j$  and  $T_j$  are real  $n$ -manifolds, intersecting only at the origin, and are tangent to  $M_j = (A_j + iI)\mathbb{R}^n$  and  $N_j = \mathbb{R}^n$  there, respectively.

Next, by following the proof of the local tubular neighborhood (see e.g. [1, Theorem 3.1] or [10, p. 78-92]) we show that in a sufficiently small neighborhood of a point  $w_0$  on a real  $k$ -submanifold  $S \subset \mathbb{R}^m$  the Taylor expansions of the squared Euclidean distance functions, respectively, to  $S$  and to the affine tangent space  $M$  to  $S$  at  $w_0$ , agree to the terms of second order. Let  $0 \in W' \subset \mathbb{R}^r$  and let  $F: W' \rightarrow W$  be a parametrization of  $S$  in a neighborhood  $W$  of a point  $w_0 \in \mathbb{R}^m$ , and such that  $F(0) = w_0$ . By the smoothness of  $S$  there exists orthonormal vector fields  $(v_1, \dots, v_{m-k}): W \cap S \rightarrow \mathbb{R}^m$ , spanning the normal space to  $S$ . We set  $\Theta: W' \times \mathbb{R}^{m-k} \rightarrow W$ ,  $\Theta(\mu, \nu) = F(\mu) + \sum_{j=1}^{m-k} \nu_j v_j(\mu)$ . Since the rank of the Jaccobian  $J(\Theta)_{(0,0)}$  is maximal, by the implicit mapping theorem  $\Theta$  is a smooth diffeomorphism in a small neighborhood of the origin. Let  $\Phi(w) = (\mu(w), \nu(w))$  be its smooth inverse, defined in a small neighborhood  $\tilde{W}$  of  $w_0$ . Observe that the nearest point on  $S$  for any  $w \in \tilde{W}$  is  $F(\mu(w))$  and the squared Euclidean distance from  $w$  to  $S$  is  $d_S(w) = |\nu(w)|^2 = \sum_{j=1}^{m-k} \nu_j^2(w)$ . Since  $d_S(w) = 0$  for  $w \in S$  all derivatives in the tangent directions to  $S$  vanish at  $w_0$ , while the derivative of  $\nu_j$  in the direction  $v_s(w_0)$  is equal to 1 if  $j = s$  and vanishes otherwise. If now  $w = (x_1, \dots, x_m)$  denote coordinates on  $\tilde{W} \subset \mathbb{R}^m$  then by Taylor's theorem we have

$$\nu_j(w) = \nu_j(w_0) + \langle v_j(w_0), w - w_0 \rangle + \sum_{|\alpha|=3} (w - w_0)^\alpha f_\alpha(w),$$

where  $(w - w_0)^\alpha = (x_1 - x_{0,1}^{\alpha_1} \dots (x_m - x_{0,m})^{\alpha_m})^{\alpha_m}$  for a multiindex  $(\alpha_1, \dots, \alpha_m)$ , and  $f_\alpha$  is a smooth function for every multiindex  $\alpha$ . It follows that

$$d_S(w) = \sum_{j=1}^{m-k} \langle v_j(w_0), w - w_0 \rangle^2 + \sum_{|\alpha|=3} (w - w_0)^\alpha g_\alpha(w) = d_M(w) + \sum_{|\alpha|=3} (w - w_0)^\alpha g_\alpha(w),$$

where  $d_M$  is a squared Euclidean distance to the affine tangent space to  $S$  at  $w_0$  and  $g_\alpha$  is a smooth function for every multiindex  $\alpha$ .

Observe further that for any homogeneous polynomial of degree  $s$  we have

$$\begin{aligned} \det H_r^{\mathbb{C}}(P(d_{S_j}, d_{T_j})_{(x,y)}) &= \det H_r^{\mathbb{C}}(P(d_{M_j}, d_{N_j})_{(x,y)}) + \sum_{|\alpha|+|\beta| > (2s-2)r} x^\alpha y^\beta h_\alpha(x, y), \\ &= \det H_r^{\mathbb{C}}(P(d_{M_j}, d_{N_j})_{(x,y)}) + \sum_{|\alpha|+|\beta| = (2s-2)r} x^\alpha y^\beta H_\alpha(x, y), \end{aligned}$$

where the determinants of the complex Hessians are homogeneous polynomials of degree  $(2s-2)r$  in  $x, y$  and  $h_\alpha, H_\alpha$  are smooth for all multiindices  $\alpha$ , and in addition  $H_\alpha(0) = 0$ . Lemma 3.3 now furnishes a homogeneous polynomial  $P$ , satisfying (3.18), and such that  $P(d_{M_j}, d_{N_j})$  is strictly plurisubharmonic everywhere, except at the origin. Since  $H_\alpha(x, y)$  is sufficiently close to zero, provided that  $(x, y)$  is close enough to the origin, by using Lemma 4.1 we deduce that  $\rho_j = P(d_{M_j}, d_{N_j})$  is strictly plurisubharmonic everywhere sufficiently close to the origin, but not at the origin. Moreover, property (3.18) yields  $S_j \cup T_j = \{\rho_j = 0\} = \{\nabla \rho_j = 0\}$  on a sufficiently small neighborhood of the origin. After possibly shrinking  $U_j$ , we set  $\varphi_j = \rho_j \circ \psi_j: U_j \rightarrow \mathbb{R}$ , and observe that  $\varphi_j$  inherits the above properties from  $\rho_j$ .

Further, let  $\varphi_0 = d_{\tilde{Z}}$  and  $d_w$  respectively be the squared distance functions to  $\tilde{Z}$  and to  $w \in \tilde{Z}$  in  $X$ , relative to some Riemannian metric on  $X$ . It is well known that the squared distance function to a smooth totally real submanifold is strictly plurisubharmonic in a neighborhood of the submanifold (see [11, Proposition 4.1]). Therefore  $\varphi_0$  is strictly plurisubharmonic in some open neighborhood  $U_0$  of  $\tilde{S} \setminus \{p_1, \dots, p_m\}$ .

By standard patching technique we now glue functions  $\varphi_j$  for all  $j \in \{0, 1, \dots, s\}$  (see e.g. [13, Theorem 2]). First, we denote  $U = \cup_{j=0}^s U_j$  and let  $p: U \rightarrow \tilde{Z}$  be a map defined as  $p(z) = w$  if  $d_{\tilde{Z}}(z) = d_w(z)$ . The map  $p$  is well defined and smooth, provided that the sets  $U_j$  are chosen small enough (see e.g. [3]). Next, we choose a partition of unity  $\{\theta_j\}_{0 \leq j \leq s}$  subordinated to  $\{U_j \cap \tilde{Z}\}_{0 \leq j \leq s}$ , and such that for every  $j \in \{1, \dots, s\}$  the function  $\theta_j$  equals one near the point  $p_j$ . Finally, we define

$$\rho(z) = \sum_{j=0}^s \theta_j(p(z)) \varphi_j(z), \quad z \in U.$$

We see that  $\tilde{Z} = \{\rho = 0\}$  and  $\nabla \rho(z) = \sum_{j=0}^m \theta_j(r(z)) \nabla \varphi_j(z)$  for all  $z \in U$ , thus we further have

$$\mathcal{L}_{(z)}(\rho; \xi) = \sum_{j=0}^s \theta_j(z) \mathcal{L}_{(z)}(\varphi_j; \xi), \quad z \in \tilde{Z}, \quad \xi \in T_p(U).$$

After shrinking  $U$  we obtain that  $\{\nabla \rho = 0\} = \tilde{Z}$  and  $\rho$  is strictly plurisubharmonic everywhere, except at the points  $q_1, \dots, q_s$ . Again, the flow of  $-\nabla \rho_0$  yields a deformation retraction of  $\Omega_\epsilon = \{\rho_0 < \epsilon\}$  onto  $\tilde{Z}$ , and by [8, p. 180] there is no compact positive dimensional analytic subset in  $\Omega_\epsilon$  (there exists a nonconstant plurisubharmonic function  $\rho$  on  $\Omega_\epsilon$ ). By a result of Grauert (see [7, Proposition 5])  $\Omega_\epsilon$  is then Stein.  $\square$

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